Section 3.1

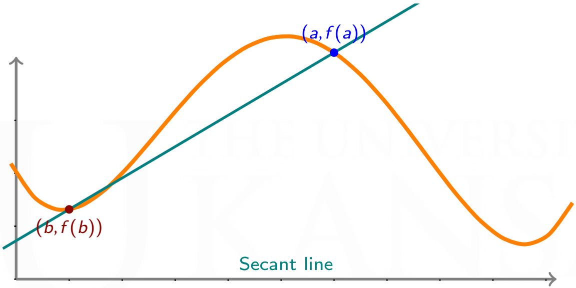
The Definition of the Derivative

(1) Tangent Lines and Tangent Slopes

(2) Instantaneous Rates of Change

(3) Differentiability

1. Graphical
2. Algebraic



Slope of a Secant Line

$$\frac{f(b)-f(a)}{b-a}$$

Slope of the Tangent Line

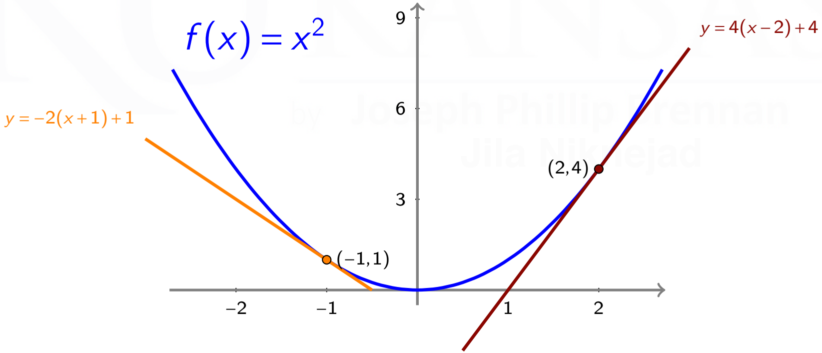
$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Derivative of a Function at a Point

The **derivative** of a function $y = f(x)$ at $x = a$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if it exists.



Let $y = f(x)$ be a function and $y = s(t)$ be a function of time, t, representing the distance traveled from a point.

Average Rate of Change over $[a, b]$

- Rate of Change

$$f_{\text{average rate}} = \frac{f(b) - f(a)}{b - a}$$

- Slope of Secant Line
- Average Velocity

$$v_{average} = \frac{s(b) - s(a)}{b - a}$$

Instantaneous Rate of Change at a

- Rate of Change - **Derivative**

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

- Slope of Tangent Line
- Instantaneous Velocity

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Tangent lines can **fail to exist.** If the derivative of $f(x)$ exists at $x = a$, then f is said to be **differentiable** at $x = a$.

Example I, Instantaneous Velocity

For the first 10 seconds after liftoff, the height of a model rocket (in meters) is given by the function

$$h(t) = t^2 + t$$

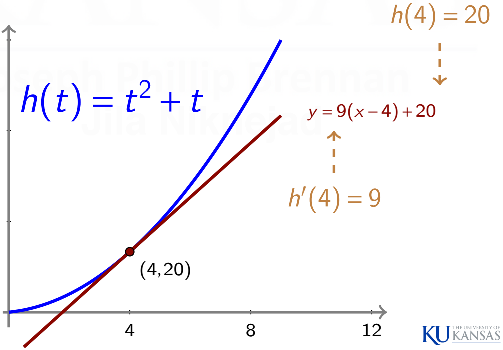
where t is the number of seconds after liftoff. How fast is the rocket traveling 4 seconds after liftoff?

Instantaneous Velocity is the Derivative of Distance

$$h'(a) = \lim_{t \to a} \frac{h(t) - h(a)}{t - a}$$

$$h'(4) = \lim_{t \to 4} \frac{t^2 + t - 20}{t - 4}$$

$$= \lim_{t \to 4} (t + 5) = 9$$



Example II, Instantaneous Rates of Change

A manufacturer produces bolts of a fabric with a fixed width. The total cost of producing x yards of this fabric is $C = f(x)$ dollars.

(i) What are the units of the derivative $f'(a)$? $\frac{\text{dollars}}{\text{yard}}$

(ii) In practical terms, what does it mean to say that $f'(1000) = 9$? The cost increases approximately by 9 dollars when the production increases from 1000 yards to 1001 yards.

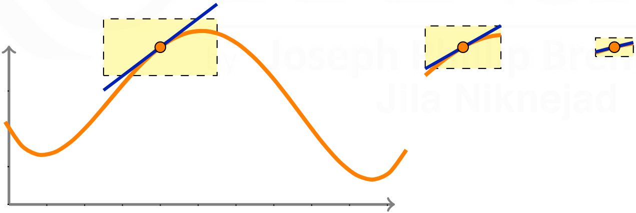
(iii) Which do you think is greatest: $f'(5)$, $f'(500)$, or $f'(5000)$? The increase in cost for an extra yard only depends on the material needed for producing a yard. Therefore, all three are the same.

What Differentiability Looks Like

Remember that $f(x)$ is said to be **differentiable** at $x = a$ if the derivative $f'(a)$ exists.

If $f'(a)$ exists, then the graph of f is **locally linear** at $x = a$.

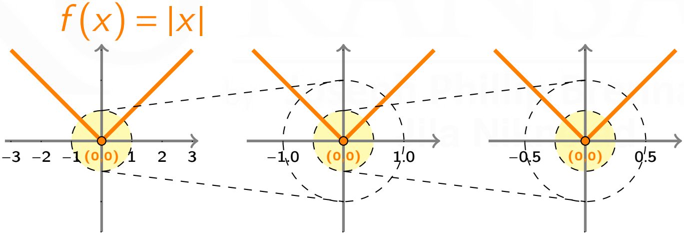
As we zoom in on the point $(a, f(a))$, the graph becomes nearly indistinguishable from its tangent line.



What Differentiability Looks Like

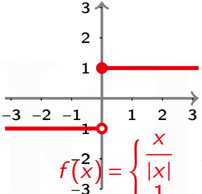
If $f'(a)$ does not exist, then the graph of f is not locally linear at $x = a$.

For example, the function $y = |x|$ is not differentiable at $x = 0$. As we zoom in toward (0,0), the corner in the graph does not disappear. So f is not locally linear at $x = 0$, and we cannot expect $f'(0)$ to exist.



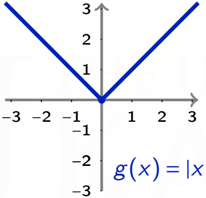
Example III, Points of Non-Differentiability

Discontinuities



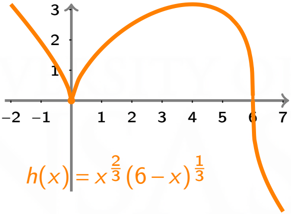
$$f(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Corners



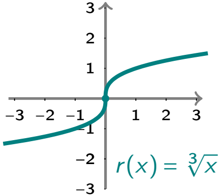
$$g(x) = |x|$$

Cusps



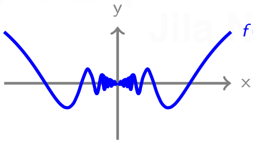
$$h(x) = x^{\frac{2}{3}}(6 - x)^{\frac{1}{3}}$$

Vertical Tangents



$$r(x) = \sqrt[3]{x}$$

Oscillating Secant Lines

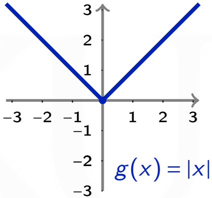


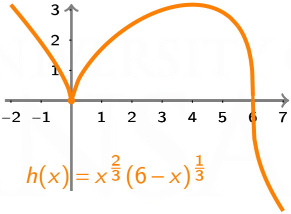
$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

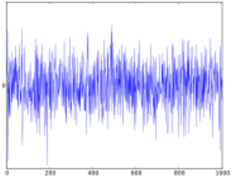
Use the following applet to see that secant lines do not settle down to a limiting position. Link

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

⚠ **Warning:** Continuity does **not** imply differentiability.







White Noise

Example IV

Each limit represents a derivative $f'(a)$. What is $f(x)$ and what is a?

(i) $\lim_{h \to 0} \frac{5^{2+h} - 25}{h}$ $\quad f(x) = 5^x \quad a = 2$

(ii) $\lim_{x \to \frac{1}{4}} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$ $\quad f(x) = \frac{1}{x} \quad a = \frac{1}{4}$

iii) $\lim_{h \to 0} \frac{(5+h)^3 - 125}{h}$ $\quad f(x) = x^3 \quad a = 5$